

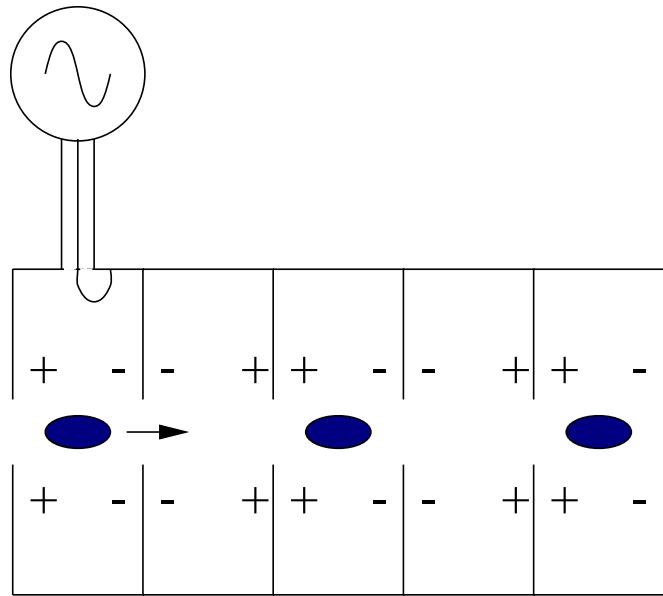
# ♪ Basics of Polarized Proton Acceleration ♫

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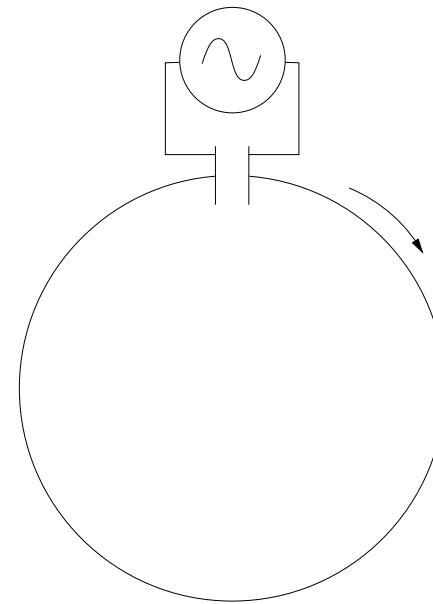
- How accelerators work
  - Particle Trajectories in Magnetic Fields.
  - Hamiltonian.
  - Hill's equation.
  - Transport and Betatron Oscillations.
  - Phase space, Liouville, & emittance.
- ~~~ Polarized beams.
  - ~~~ Simple model of the proton.
    - ~~~ Relativistic angular momentum.
    - ~~~ Relativistic spin precession. (Thomas—Frenkel Eq.)
  - ~~~ Spin precession in an accelerator ring, spin tune.
  - ~~~ Spin dynamics.
  - ~~~ Depolarizing resonances.
  - ~~~ Siberian snakes,  $\nu_{\text{spin}} = \frac{1}{2}$ .
- ~~~ The real machines: RHIC and injectors.



# Acceleration with RF cavities



Linac:  $\vec{F} = q\vec{E}(t)$ .

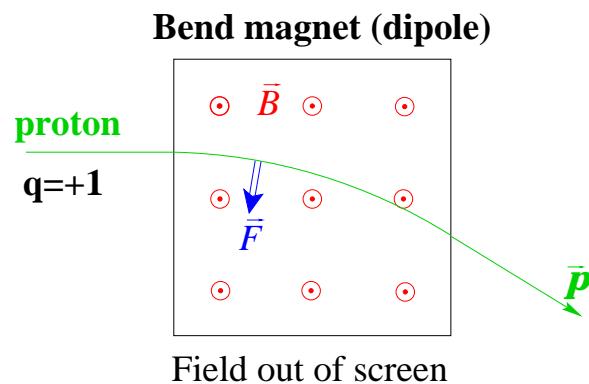


Ring with rf cavity

- Must maintain synchronism of bunch with rf phase.
- Particles oscillate in energy about the stable synchronous phase.

# Particle Trajectories in Magnetic Fields

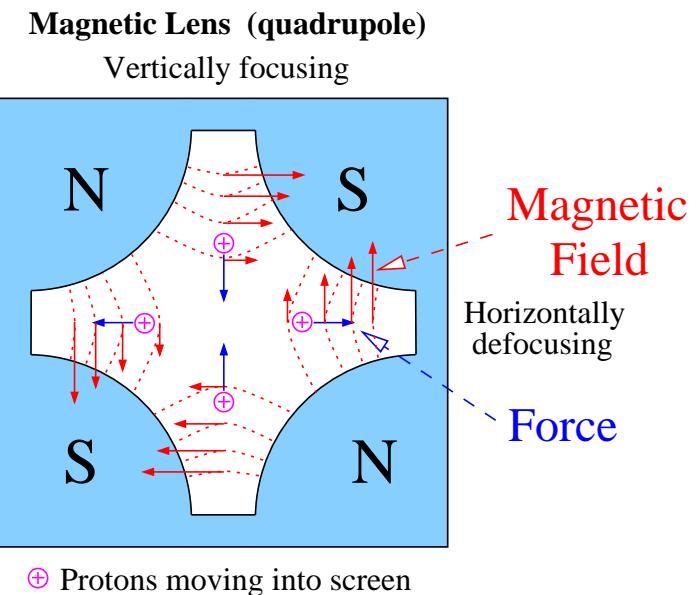
Dipole magnets bend the beam around the ring.



Charged particles are deflected by magnetic fields. Lorentz Force:

$$\vec{F} = \frac{q}{\gamma m} \vec{p} \times \vec{B}$$

Quadrupole magnets focus the beam for stability.



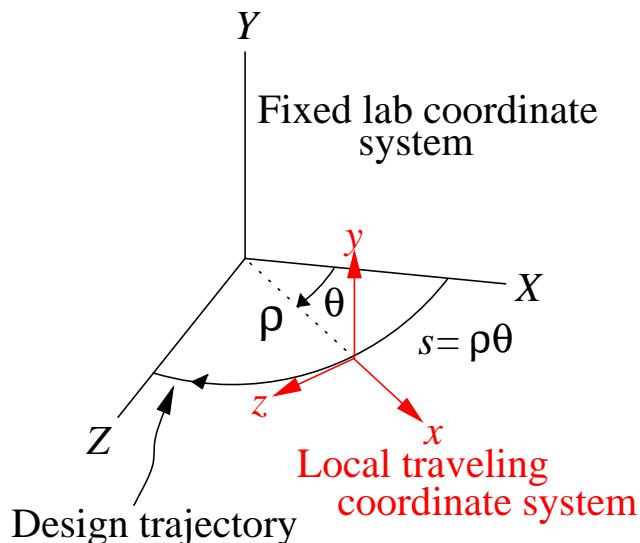
# ⌚ Hamiltonian without Spin ⌚

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$$H(X, P_X, Y, P_Y, Z, P_Z; t) = \sqrt{(\vec{P} - q\vec{A})^2 + m^2c^4} + q\phi$$

After a bunch of canonical transformations and  $\phi = 0$ ,  $\vec{A} = (0, 0, A_s)$ :

$$\mathcal{H}(x, x', y, y', z, \delta p/p_0; s) \simeq -\frac{q}{p_0}A_s - \left(1 + \frac{x}{\rho}\right) \left(1 + \frac{\delta p}{p_0} - \frac{1}{2}(x'^2 + y'^2) + \dots\right)$$



$$\begin{aligned}\rho &= \frac{p}{qB_\perp} \\ x' &= \frac{dx}{ds} \\ y' &= \frac{dy}{ds}\end{aligned}$$

Paraxial approx.:  $|x'|, |y'| \ll 1$

# ♪ Hill's Equations ♪

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$$x'' + k_x(s)x = \frac{\delta}{\rho(s)},$$

$$y'' + k_y(s)y = 0,$$

$$\text{with } \delta = \frac{\delta p}{p_0}.$$

For quadrupoles:

$$k_x = \frac{q}{p} \frac{\partial B_y}{\partial x}$$

$$k_y = -\frac{q}{p} \frac{\partial B_y}{\partial x}$$

Harmonic oscillator with periodic spring constant.

Periodic conditions:  $k_j(s + L) = k_j(s)$ ,  $\rho(s + L) = \rho(s)$

where  $L$  is length of periodic cell.

- Horizontal motion has inhomogeneous dispersion term.
  - Ignore it for now.

# ♪ Solutions to Hill's Equation ♪

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Use Floquet's (Block's) Theorem  $\Rightarrow$

Quasi-periodic solutions of form:

$$x(s) = \sqrt{\mathcal{W}\beta(s)} \cos(\psi(s)), \quad \text{with}$$
$$\psi'(s) = \frac{1}{\beta(s)}.$$

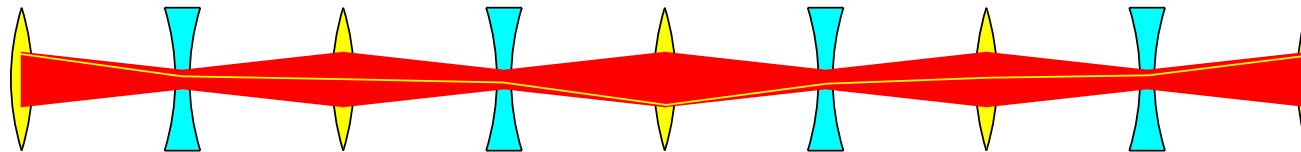
Periodicity of  $\beta$ -function:  $\beta(s + L) = \beta(s)$ .

Note: In general  $\psi(s + L) \neq \psi(s) + n2\pi$ . Resonances are bad!

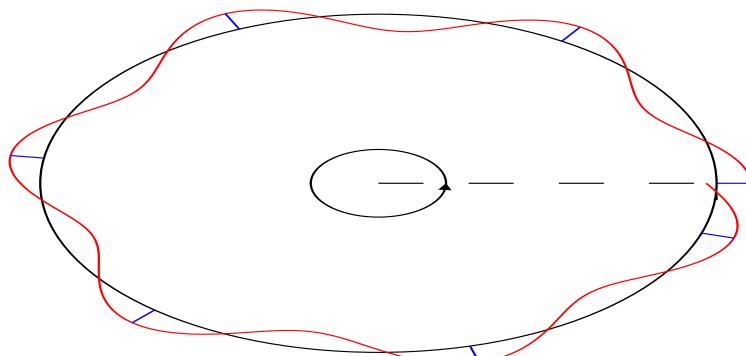
$$x'(s) = -\sqrt{\frac{\mathcal{W}}{\beta}} (\alpha \cos \psi + \sin \psi),$$

with  $\alpha = -\frac{1}{2}\beta'$ .

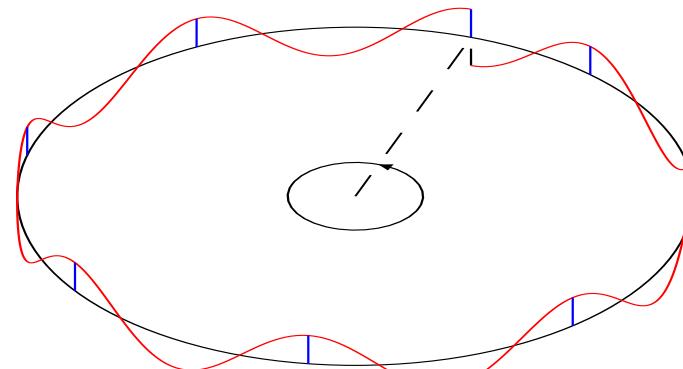
# Transport and Betatron Oscillations



Alternate focusing and defocusing lenses for stability.



Horizontal Betatron Oscillation  
with tune:  $Q_h = 6.3$ ,  
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation  
with tune:  $Q_v = 7.5$ ,  
i.e., 7.5 oscillations per turn.

# Courant–Snyder Invariant

For a particular trajectory with initial conditions:

- Solve for  $\sin \psi$  and  $\cos \psi$  from equations for  $x$  and  $x'$ .
- Use  $\sin^2 \psi + \cos^2 \psi = 1$  to get an invariant:

$$\mathcal{W} = \frac{1}{\beta} [y^2 + (\alpha y + \beta y')^2] \quad (1)$$

- Functions of  $s$ :  $y(s)$ ,  $y'(s)$ ,  $\beta(s)$ ,  $\alpha(s)$ .  $(\beta$  and  $\alpha$  are periodic.)
- Eq. (1) is the equation for an ellipse.
  - Area of ellipse =  $\pi \mathcal{W}$ .
- Beam envelope:  $\pm \sqrt{\beta(s)} \epsilon$ 
  - $\pi \epsilon$  is the rms emittance

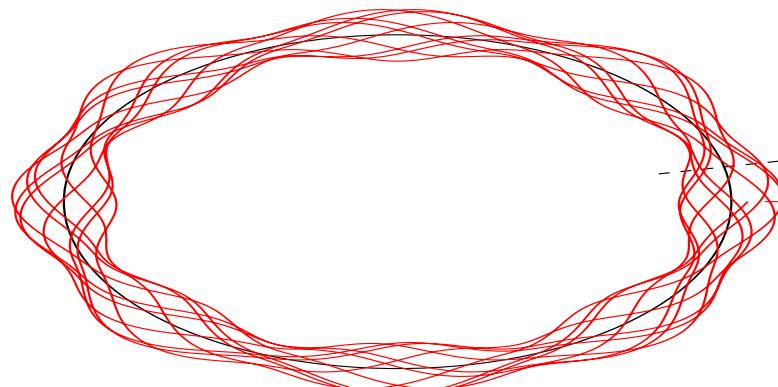
# Liouville's Theorem

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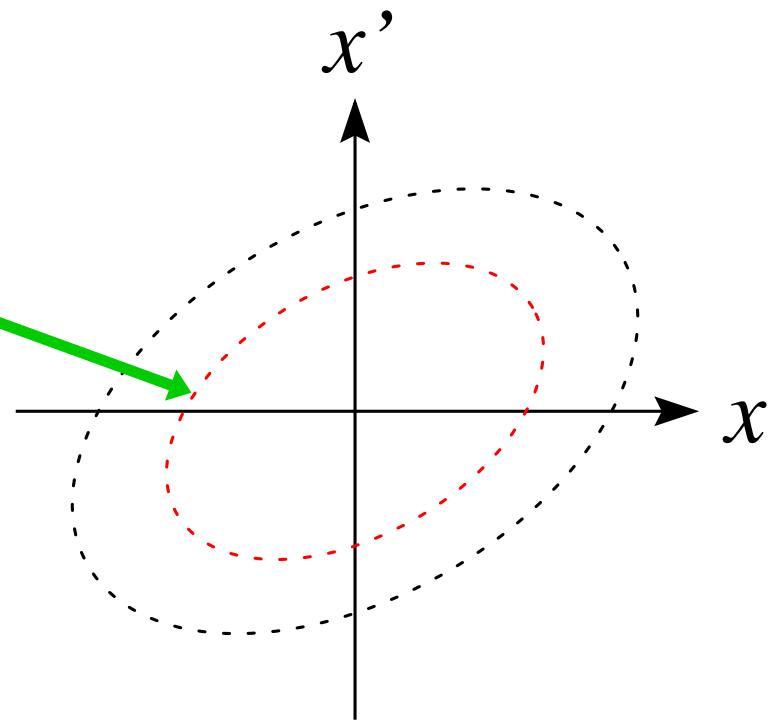
- Most beams have a low enough density, so that we ignore hard collisions between particles.
  - Thus we can use a 6d phase space rather than a  $6N-d$  phase space.
- In the phase space of coordinates and their corresponding canonical momenta, the phase flow of the particle trajectories evolves so that the volumes of differential volume elements are preserved.
  - In other words, the Jacobian determinant is 1.
- Emittance is the area of the projection of the beam's phase-space volume onto a particular  $(x_i, P_i)$  plane.

# ♪ 2d Phase Space Plots ♪

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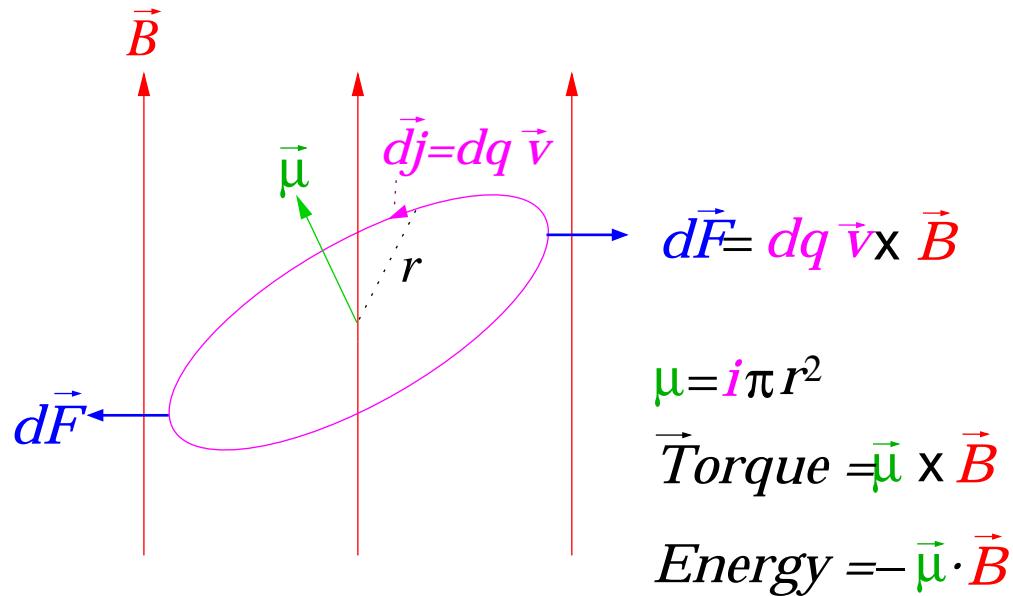


Horizontal Betatron Oscillation  
with tune:  $Q_x = 3.28$ ,  
tracked through 10 turns  
with 8 periodic cells.



Poincaré plot of proton on successive turns for one location in the ring.

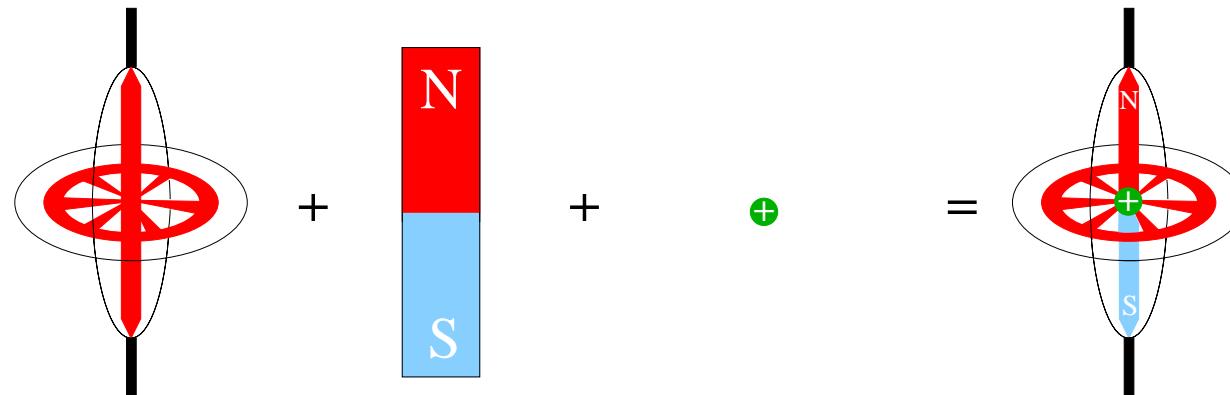
# ♪ Torque on Classical Magnetic Moment ♪



~ Semiclassical model:

- The spin  $\vec{S}$  has a constant magnitude in the rest frame.
- The magnetic moment  $\vec{\mu} \propto \vec{S}$ .
  - $\vec{\mu}$  has a constant magnitude in the rest frame.  
(Sort of like a loop of infinite inductance.)

# Simple Model of Proton



Gyroscope + Bar magnet + Charge = "proton"

Magnetic  
Spin      Dipole  
            Moment

Polarization: Average spin of the ensemble of protons.

# Relativistic Angular Momentum

Energy-momentum tensor (à la Weinberg)

$$T^{\alpha\beta}(x) = T^{\beta\alpha}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(x - x_n(t))$$

For isolated system

$$\frac{\partial}{\partial x^\alpha} T^{\alpha\beta} = 0.$$

Define 4d analogue of  $\vec{r} \times \vec{p}$ :

$$M^{\alpha\beta\gamma} = x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}$$
$$J^{\alpha\beta} = \int M^{0\alpha\beta} d^3x = \int x^\alpha T^{\beta 0} - x^\beta T^{\alpha 0} d^3x$$

Spin (intrinsic angular momentum):

$$S_\alpha = \frac{1}{2c} \epsilon_{\alpha\beta\gamma\delta} J^{\beta\gamma} u^\delta, \quad \text{proper velocity: } u^\delta = \frac{dx^\delta}{d\tau}.$$



For a particle at rest with CM at rest at the origin:

$$J^{\diamond\mu\nu} : \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_z^\diamond & -S_y^\diamond \\ 0 & -S_z^\diamond & 0 & S_x^\diamond \\ 0 & S_y^\diamond & -S_x^\diamond & 0 \end{pmatrix}, \quad (\vec{J}^\diamond = \vec{S}^\diamond)$$

Boost along  $z$ :

$$J^{\mu\nu} : \begin{pmatrix} 0 & \gamma\beta S_y^\diamond & -\gamma\beta S_x^\diamond & 0 \\ -\gamma\beta S_y^\diamond & 0 & S_z^\diamond & -\gamma S_y^\diamond \\ \gamma\beta S_x^\diamond & -S_z^\diamond & 0 & \gamma S_x^\diamond \\ 0 & \gamma S_y^\diamond & -\gamma S_x^\diamond & 0 \end{pmatrix}, \quad \Rightarrow \quad \vec{J} = \begin{pmatrix} \gamma S_x^\diamond \\ \gamma S_y^\diamond \\ S_z^\diamond \end{pmatrix}$$

$$S^\mu : \begin{pmatrix} \gamma\beta S_z^\diamond \\ S_x^\diamond \\ S_y^\diamond \\ \gamma S_z^\diamond \end{pmatrix}, \quad \Rightarrow \quad \vec{S} = \begin{pmatrix} S_x^\diamond \\ S_y^\diamond \\ \gamma S_z^\diamond \end{pmatrix}, \quad S^0 = \vec{\beta} \cdot \vec{S}$$

$$\vec{J} - \vec{S} = \begin{pmatrix} (\gamma - 1)S_x^\diamond \\ (\gamma - 1)S_y^\diamond \\ (1 - \gamma)S_z^\diamond \end{pmatrix}$$

# ♪ Center-of-Mass shift ♪

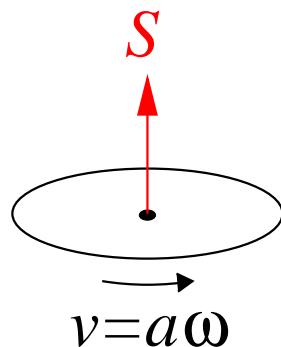
$$\vec{r}_{\text{CM}} \times \vec{p}_{\text{CM}} = (\vec{J} - \vec{S})_{\perp}$$

$$\gamma \beta m c (-x_{\text{CM}} \hat{y} + y_{\text{CM}} \hat{x}) = (\gamma - 1) \vec{S}_{\perp}^{\diamond}$$

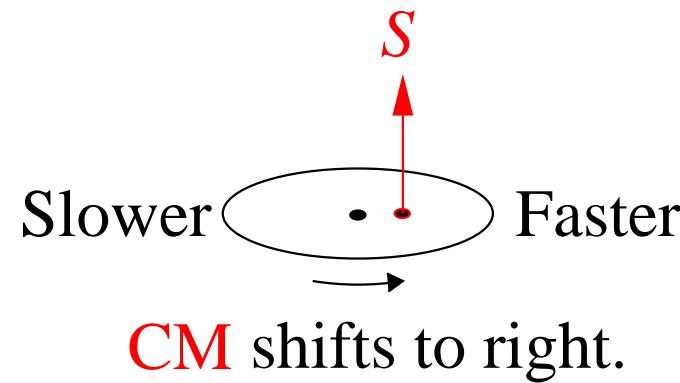
$$\gamma \beta m c (\vec{x}_{\text{CM}} + \vec{y}_{\text{CM}}) = (\gamma - 1) \hat{z} \times \vec{S}_{\perp}^{\diamond}$$

$$\vec{r}_{\perp \text{CM}} = \frac{\gamma}{\gamma + 1} \frac{\vec{\beta} \times \vec{S}}{mc}$$

CM at rest.



Boost into screen



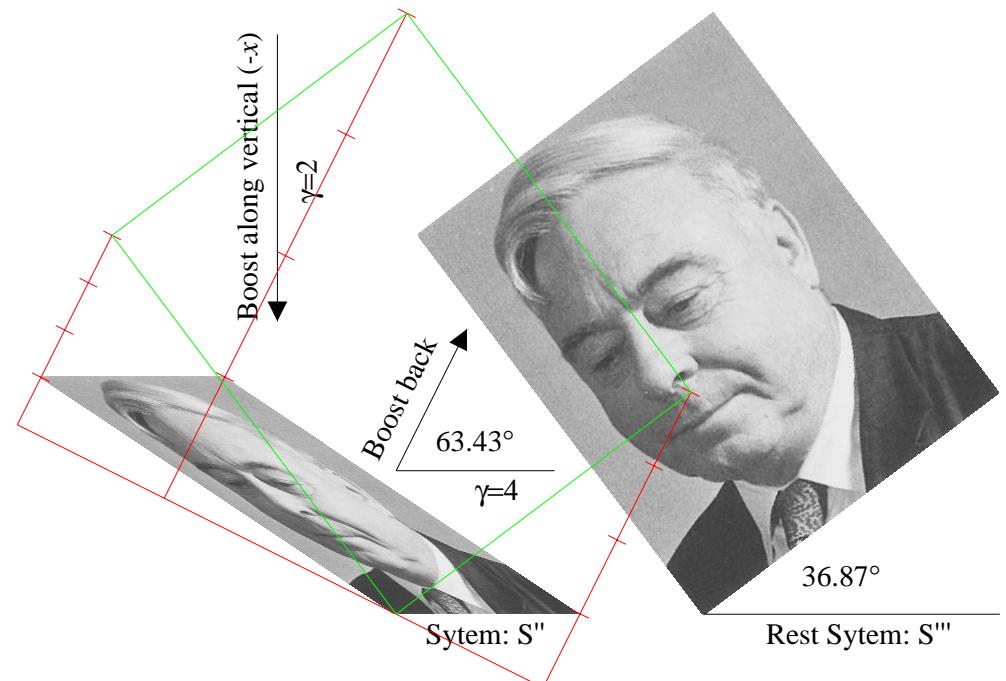
# Thomas Precession

1. Boost observer to left.
  2. Boost observer downward.
  3. Boost back to rest.
- Net rotation of rest frame.



System: S'

Rest System: S



SPIN 2002 Tutorial  
Waldo MacKay 8, Sept., 2002

# ♪ Thomas—Frenkel (BMT) Equation ♪

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In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

$$\frac{d\vec{S}^\diamond}{dt} = \frac{q}{\gamma m} \vec{S}^\diamond \times \left[ (1 + G\gamma) \vec{B}_\perp + (1 + G) \vec{B}_\parallel + \left( G\gamma + \frac{\gamma}{\gamma + 1} \right) \frac{\vec{E} \times \vec{v}}{c^2} \right].$$

This is a mixed description:  $t$ ,  $\vec{B}$ , and  $\vec{E}$  in the lab frame, but spin  $\vec{S}^\diamond$  in local rest frame of the proton.

$$G = \frac{g - 2}{2} = 1.7928, \quad \gamma = \frac{\text{Energy}}{mc^2}.$$



# ♪ Thomas—Frenkel (BMT) Equation ♪

---

In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

$$\text{Torque : } \frac{d\vec{S}^\diamond}{dt} = \frac{q}{\gamma m} \vec{S}^\diamond \times \left[ (1 + G\gamma) \vec{B}_\perp + (1 + G) \vec{B}_\parallel \right] \quad \text{TF}$$

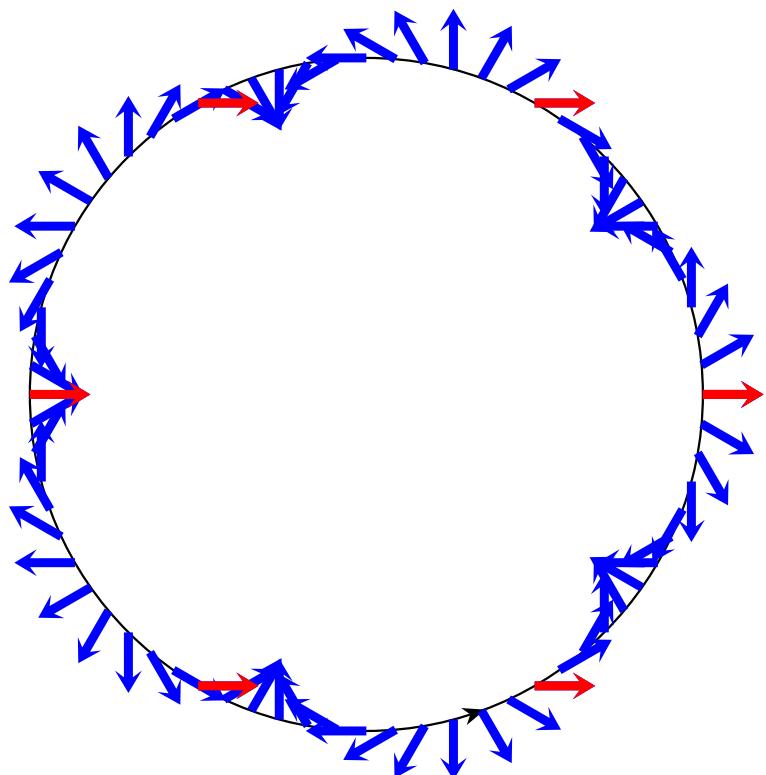
$$\text{Force : } \frac{d\vec{p}}{dt} = \frac{q}{\gamma m} \vec{p} \times \vec{B}_\perp \quad \text{Lorentz}$$

(This is a mixed description:  $t$ , and  $\vec{B}$  in the lab frame, but spin  $\vec{S}^\diamond$  in local rest frame of the proton.)

$$G = \frac{g - 2}{2} = 1.7928, \quad \gamma = \frac{\text{Energy}}{mc^2}.$$

# ♪ Spin Precession in a Ring ♪

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Example with 6 precessions of spin in one turn:

$$G\gamma + 1 = 6.$$

Spin tune: number of precessions per turn relative to beam's direction.

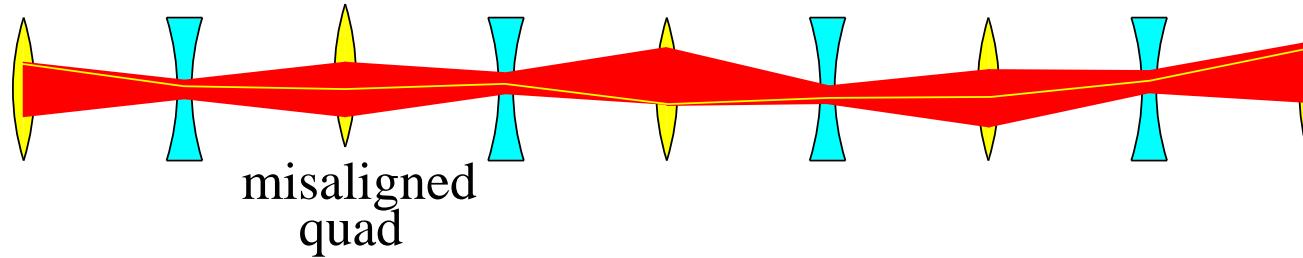
So we subtract one:

$$\nu_{\text{spin}} = G\gamma \propto \text{energy},$$

i.e., 5 in this example.

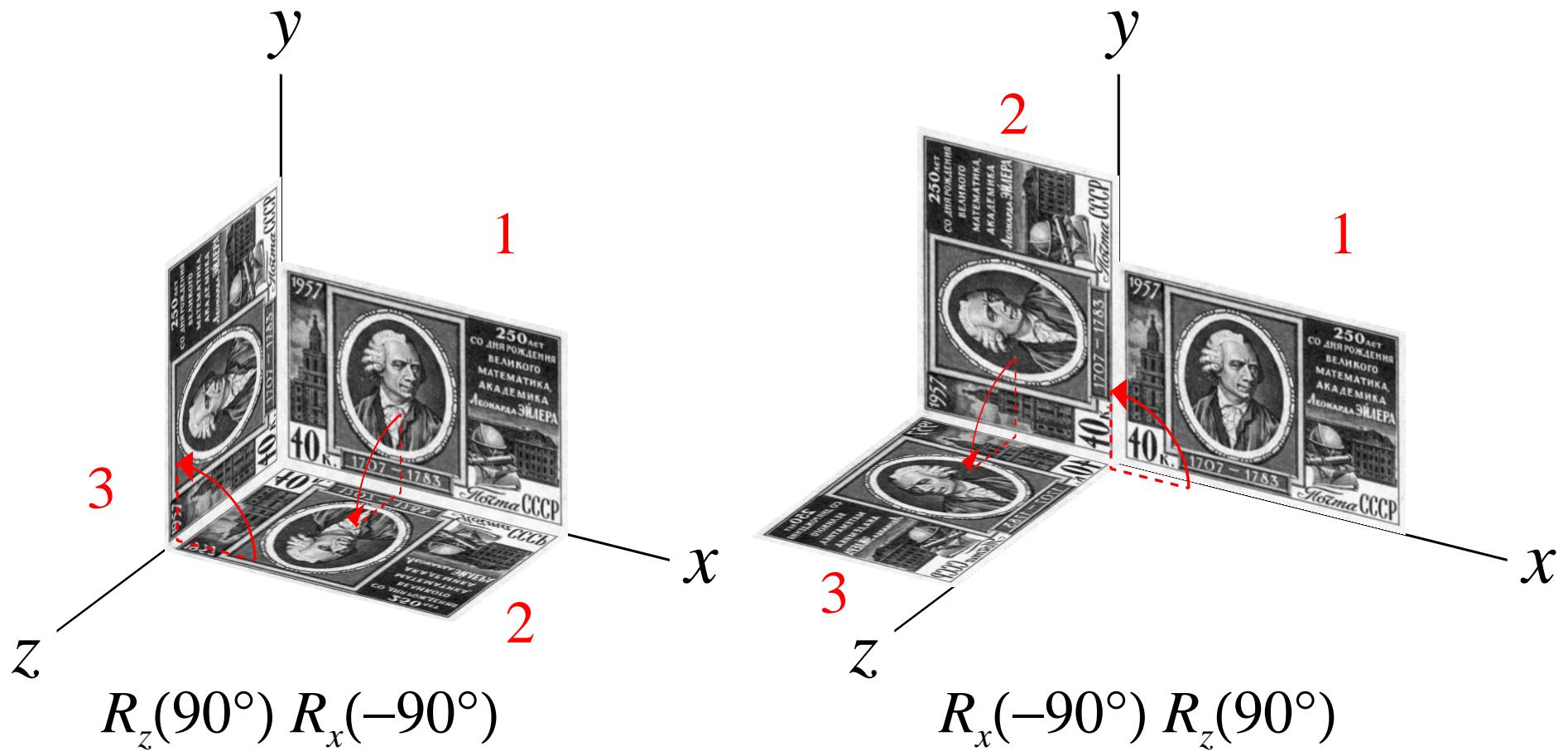
# ♪ Misalignments or Imperfections ♪

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- A misaligned quadrupole creates a steering error which propagates through the lattice.
- For an accelerator ring, this shifts the closed orbit away from the design trajectory.
- If the misalignment is vertical, then the design trajectory will have a periodic set of small vertical bends interspersed with the normal horizontal bends of the bending magnets.
- This leads to an integer resonance condition for the spin tune.

# ♪ In general, rotations don't commute. ♪



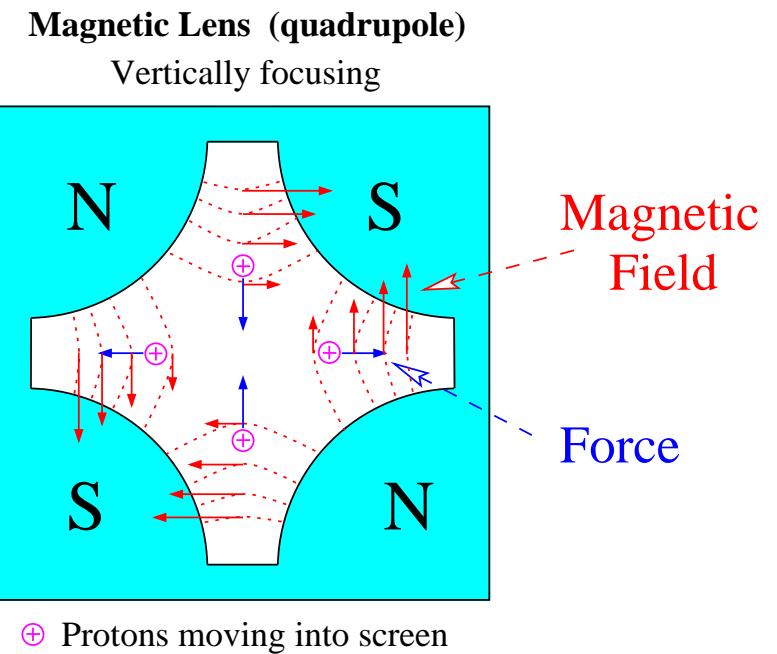
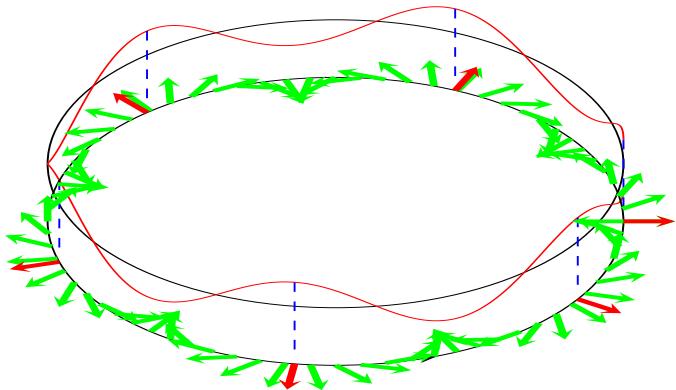
# Depolarizing Resonances

Simple Resonance Condition:

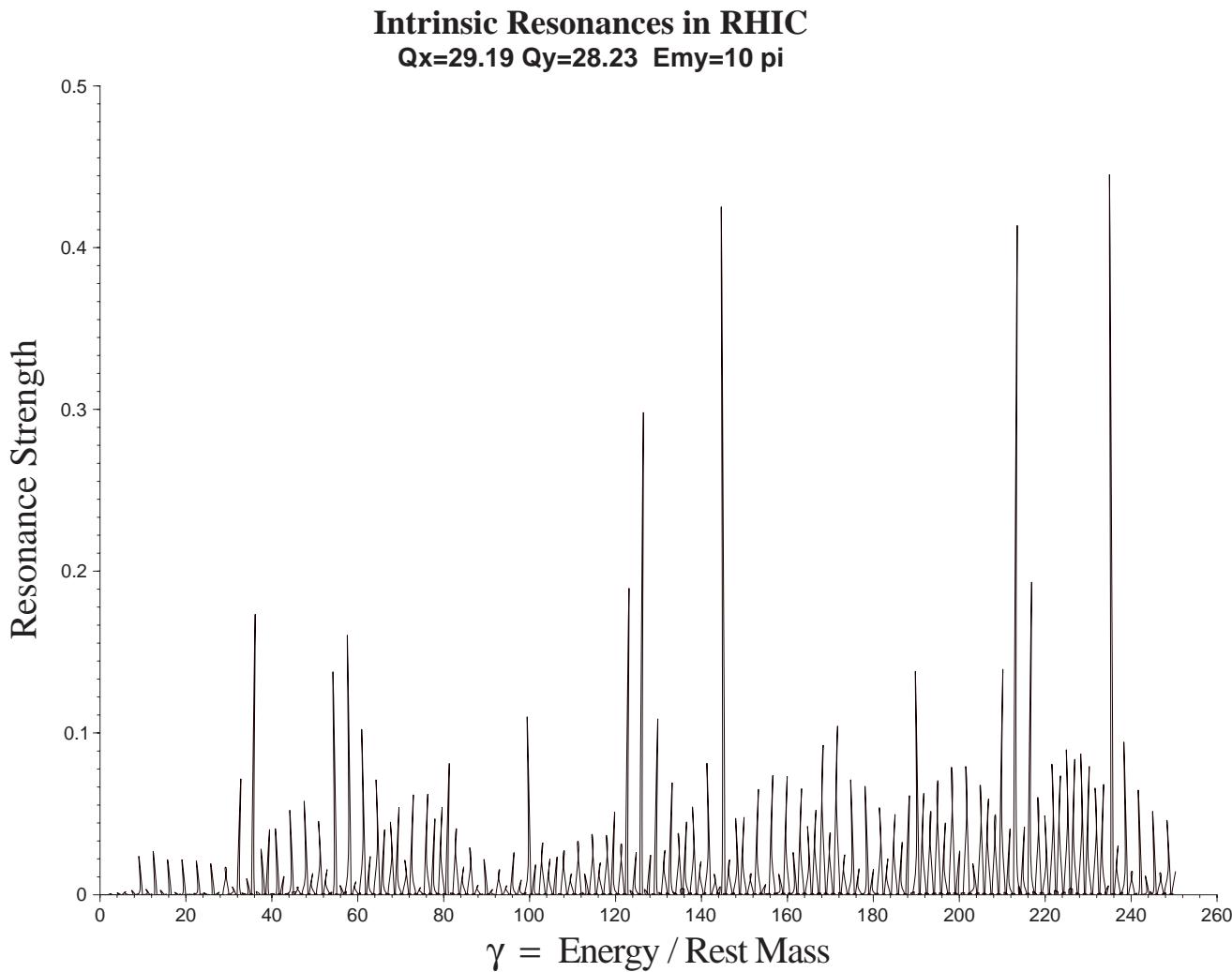
$$\nu_{\text{spin}} = N \quad + \quad N_v Q_v,$$

(imperfection)      (intrinsic)

where  $N$  and  $N_v$  are integers.



# Depolarizing Resonances

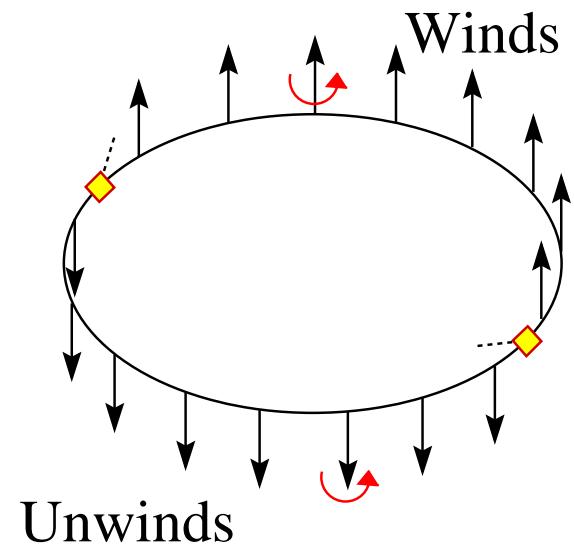


Will depolarize beam during acceleration.

**Solution: Snakes**

# Snake Charming

- 2 snakes: spin is up in one half of the ring, and down in the other half.
- Spin tune:  $\nu_{\text{spin}} = \frac{1}{2}$   
(It's energy independent.)
- “The unwanted precession which happens to the spin in one half of the ring is unwound in the other half.”



# Hamiltonian with Spin

---

(Here I drop the “ $\diamond$ ” superscript on  $\vec{S}$ .)

$$\begin{aligned}\frac{d\vec{S}}{dt} &= \vec{W} \times \vec{S} \\ H(\vec{q}, \vec{P}, \vec{S}; s) &= \mathcal{H}_{\text{orb}} + \mathcal{H}_{\text{spin}} \\ &= \mathcal{H}_{\text{orb}} + \vec{W} \cdot \vec{S} + O(\hbar^2)\end{aligned}$$

Group symmetries:

- Orbital motion without spin:  $\text{Sp}(6, r)$ .
- Spin by itself:  $\text{SU}(2, c) \cong \text{SO}(3, r)$  (homomorphic).
- Full blown symmetry:  $\text{Sp}(6, r) \oplus \text{SU}(2, c)$ .
  - Spin dependence on orbit (Thomas-Frenkel).
  - Orbit dependence on spin (Stern-Gerlach Force)—Usually ignored.

# § Representation of Rotations §

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SU(2) with usual spinor notation:

Pauli matrices:  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

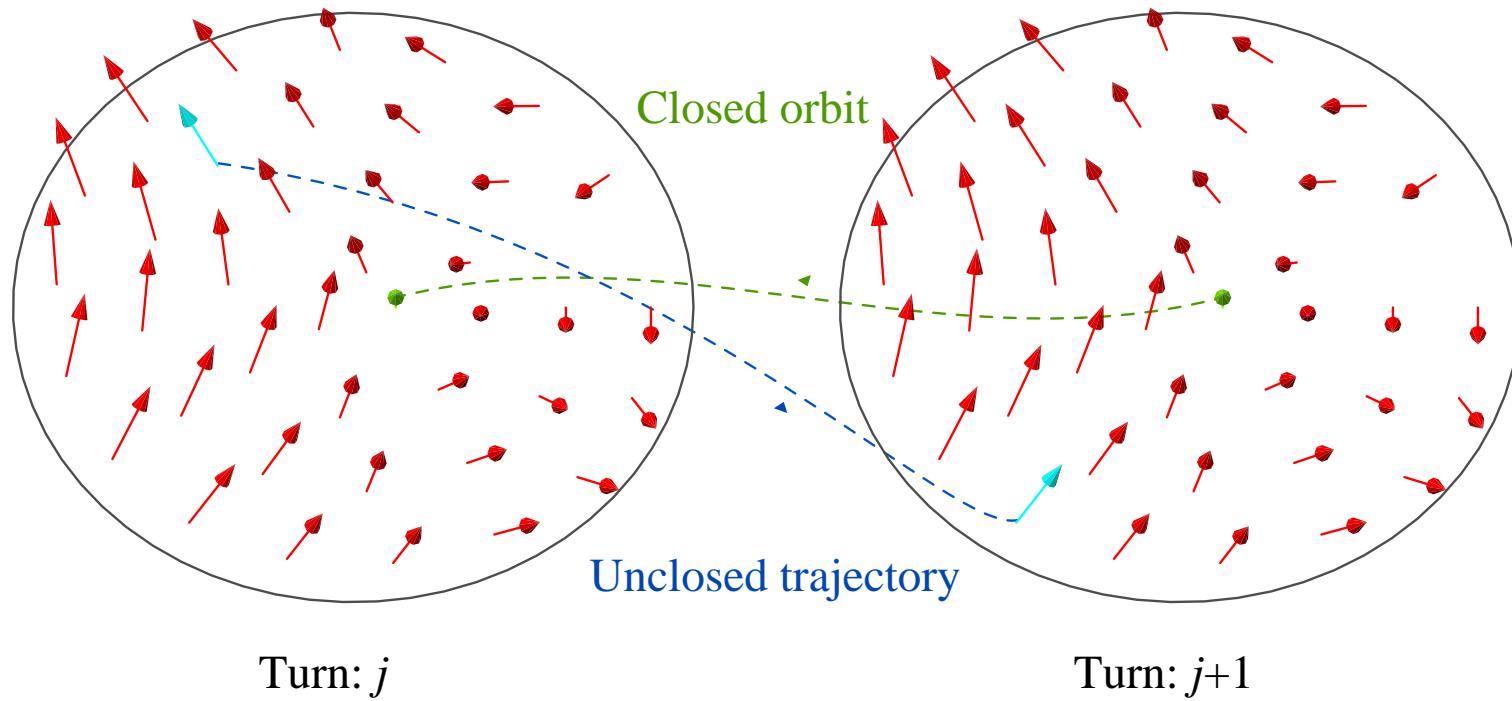
$$\mathbf{R}_{\hat{n}}(\theta) = e^{i \hat{n} \cdot \vec{\sigma} \theta / 2} = \begin{pmatrix} \cos \frac{\theta}{2} + i n_z \sin \frac{\theta}{2} & (n_y + i n_x) \sin \frac{\theta}{2} \\ (-n_y + i n_x) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} - i n_z \sin \frac{\theta}{2} \end{pmatrix}.$$

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SO(3) :

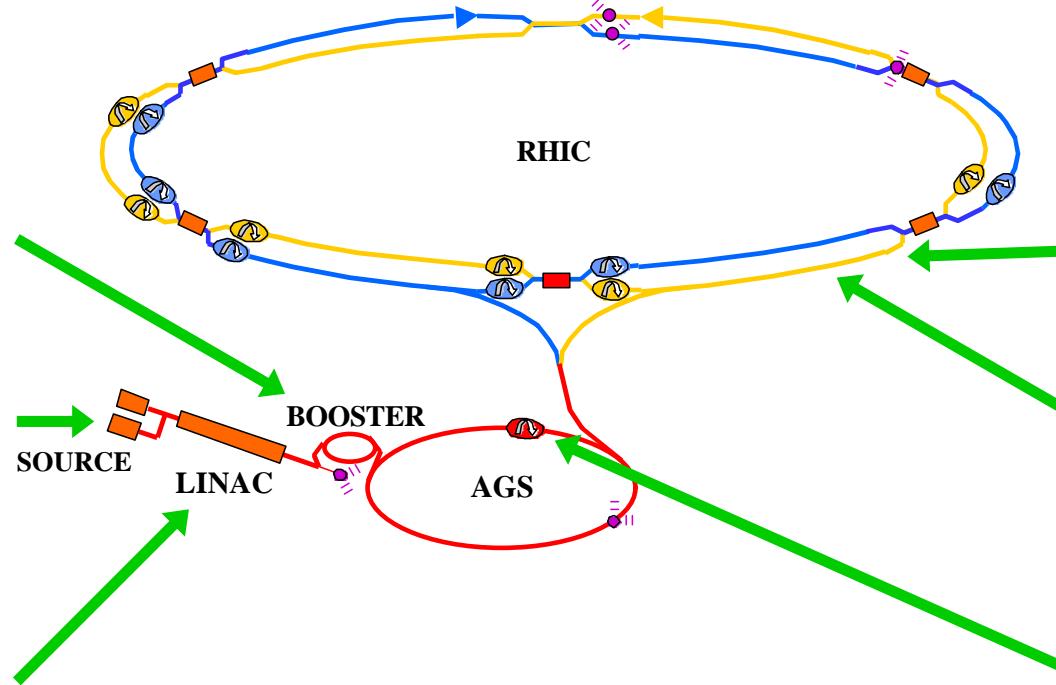
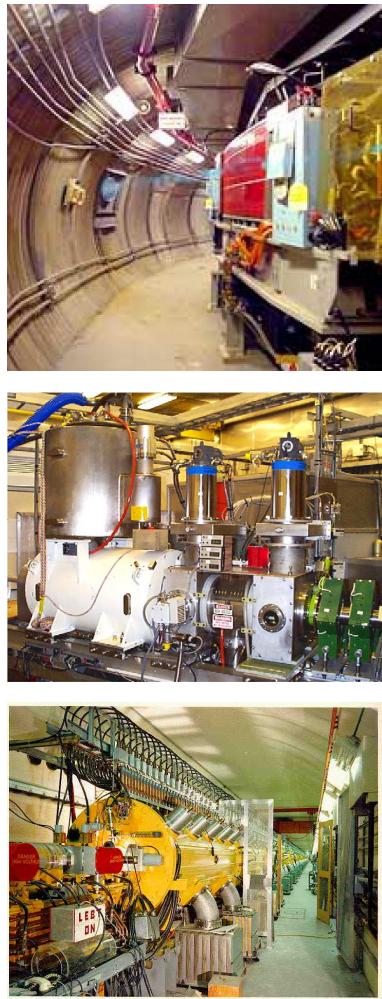
$$\mathbf{R}_{\hat{n}}(\theta) = \mathbf{I} \cos \theta + \begin{pmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{pmatrix} \sin \theta + \begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{pmatrix} (1 - \cos \theta).$$

# Invariant Spin Field



- For the closed orbit:  $\vec{n}_0(s) = \vec{n}_0(s + L)$ ,  
with  $\vec{q}_0(s) = \vec{q}_0(s + L)$  and  $\vec{P}_0(s) = \vec{P}_0(s + L)$ .
- For other locations in phase space:  $\vec{n}(\vec{q}, \vec{P}, s) = \vec{n}(\vec{q}, \vec{P}, s + L)$ ,  
even though in general  $q(s + L) \neq q(s)$  and  $P(s + L) \neq P(s)$ .

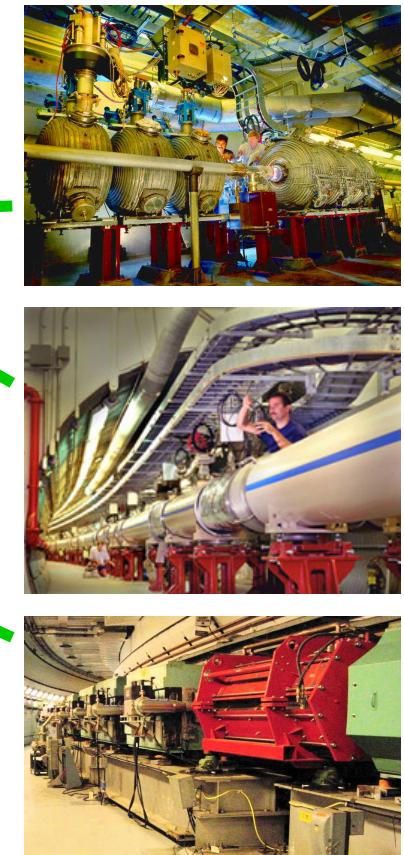
# Accelerator Complex (Pol. Protons)



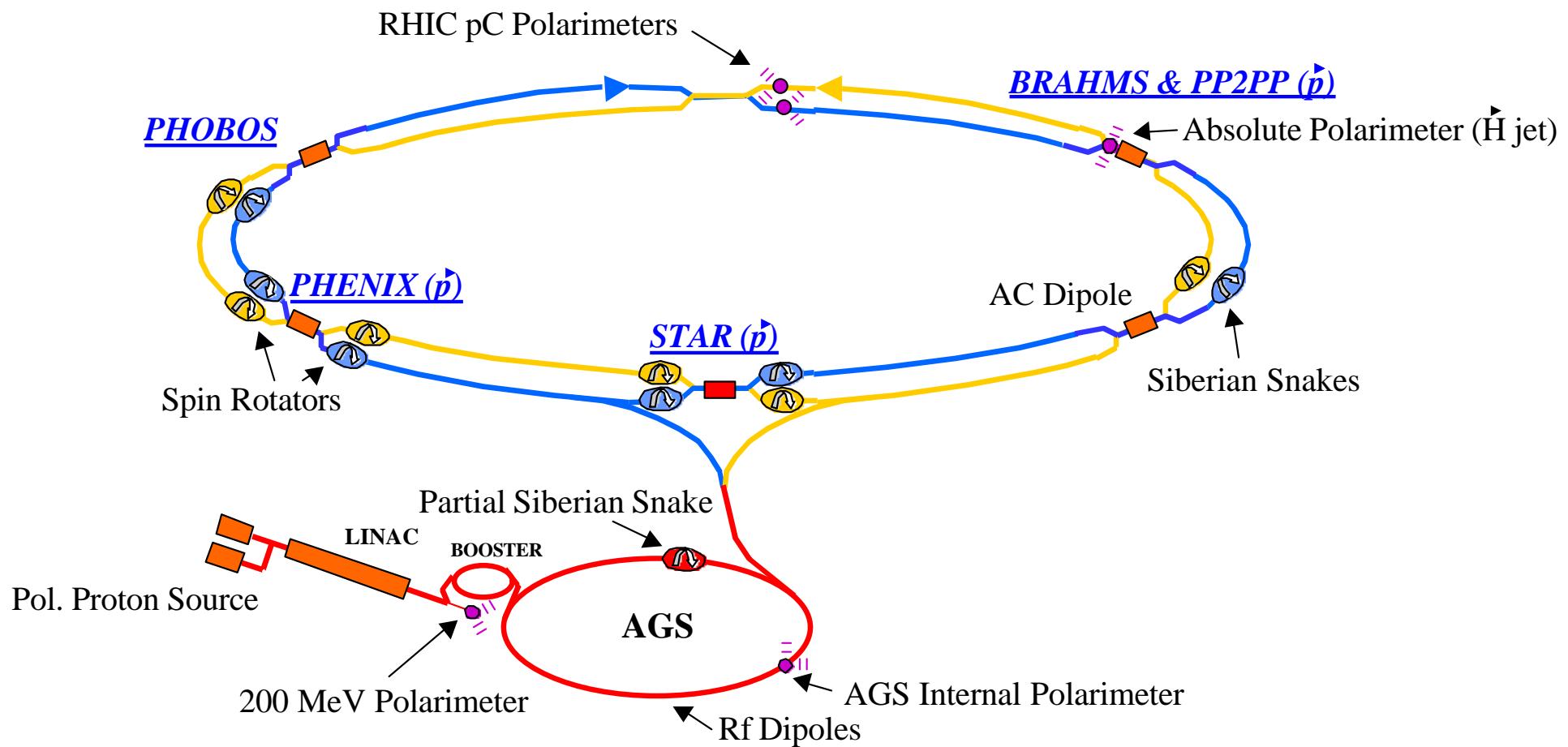
LINAC: Linear Accelerator

AGS: Alternating Gradient Synchrotron

RHIC: Relativistic Heavy Ion Collider



# Accelerator Complex for Protons



# ♪ High Intensity Polarized H<sup>-</sup> Source ♪



KEK OPPIS\*  
upgraded at TRIUMF  
 $70 \rightarrow 80\%$  Polarization  
 $15 \times 10^{11}$  protons/pulse  
at source  
 $6 \times 10^{11}$  protons/pulse  
at end of LINAC

\*Optically Pumped Polarized Ion Source

# Optically Pumped Polarized Ion Source

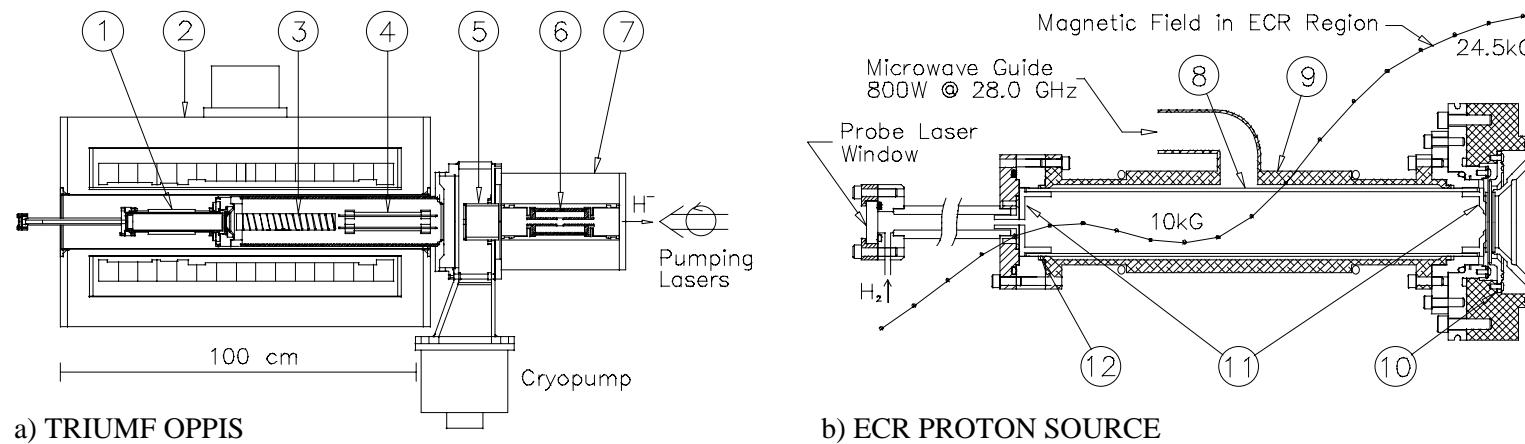
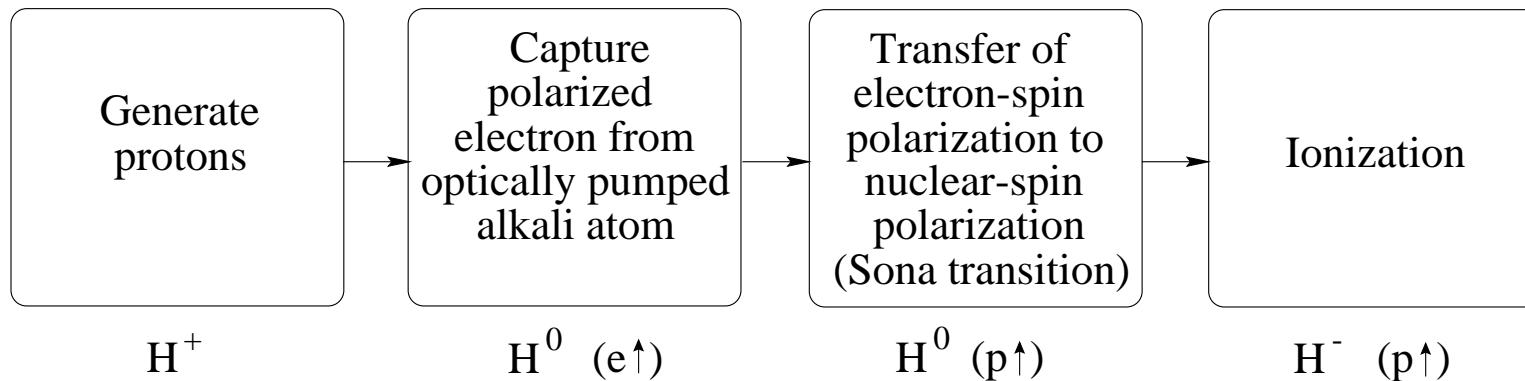
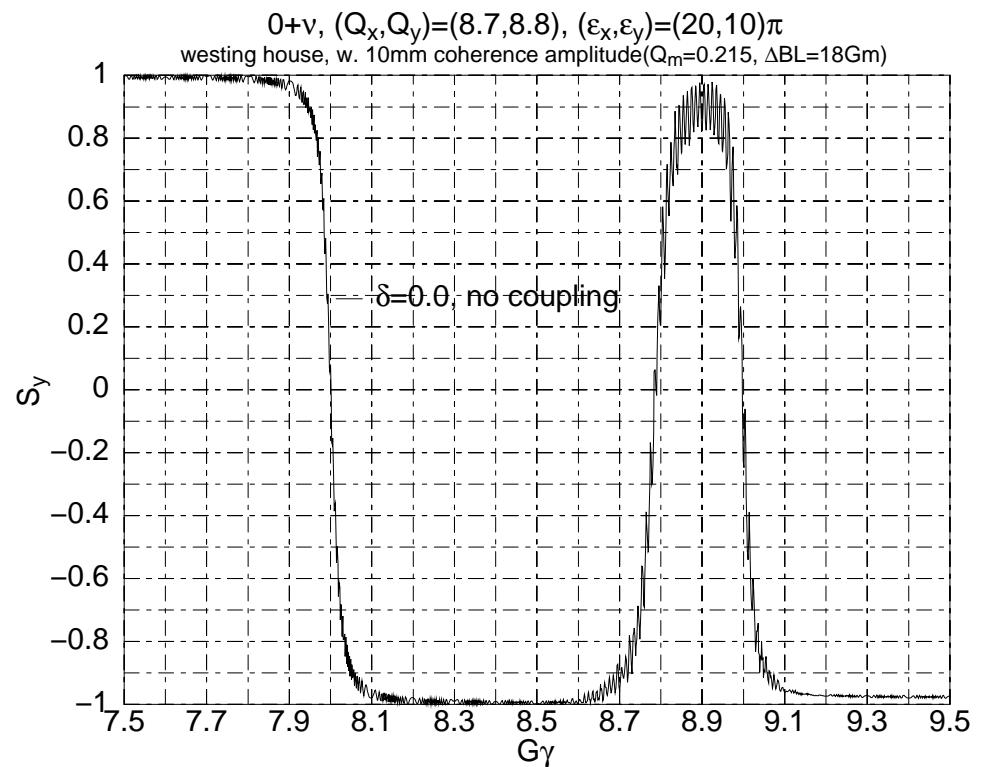
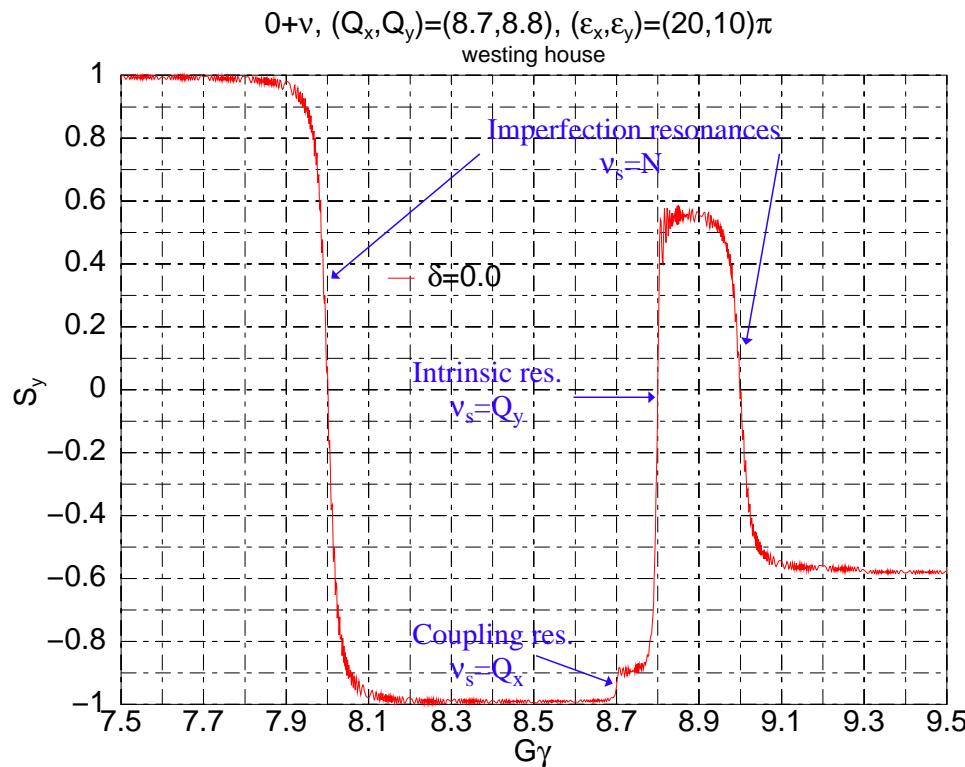


Fig. 1. 1) ECR Proton Source, 2) Superconducting Solenoid, 3) Optically-Pumped Rb Cell, 4) Deflection Plates, 5) Sona Transition Region, 6) Ionizer Cell, 7) Ionizer Solenoid, 8) Quartz Tube, 9) ECR Cavity, 10) Three Grid Extraction System, 11) Boron-Nitride End Cups, 12) Indium Seals.



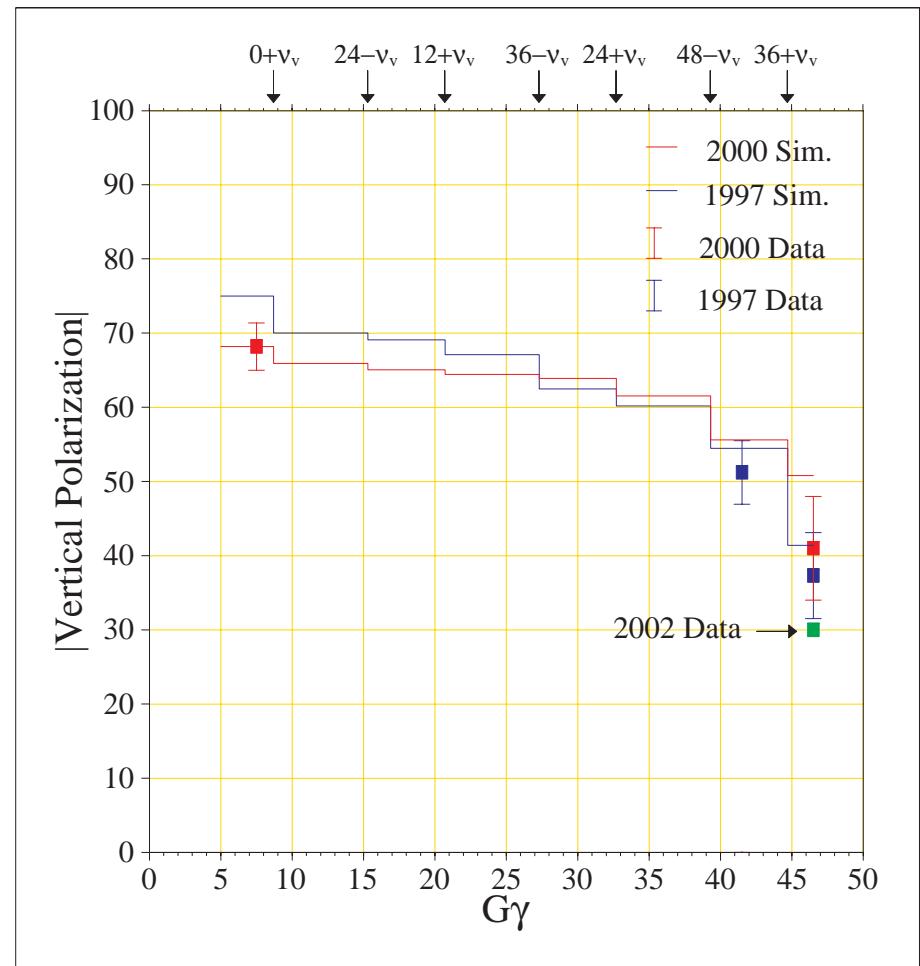
# Resonance Crossing in AGS



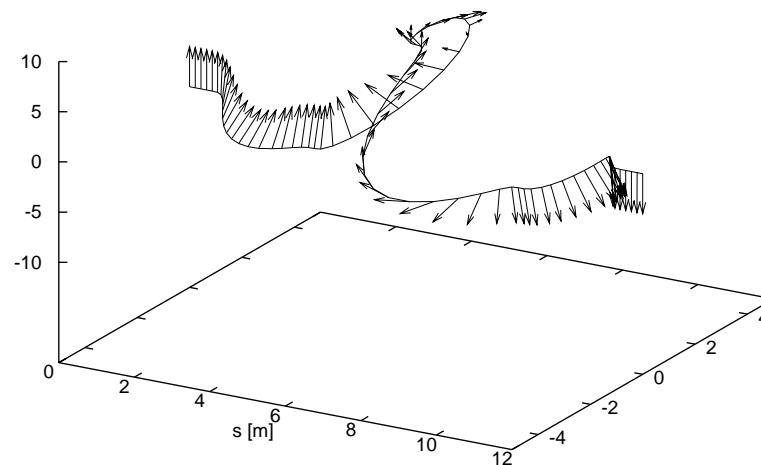
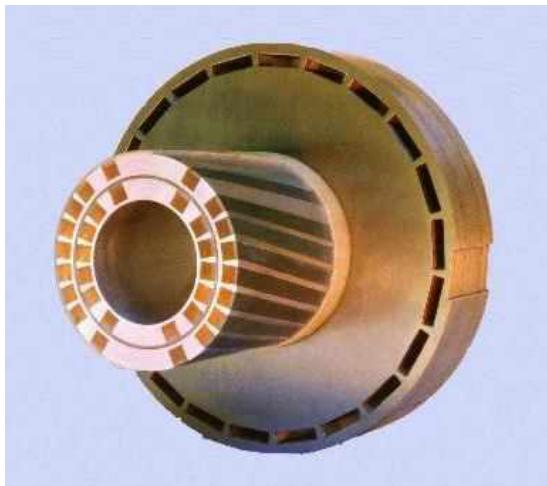
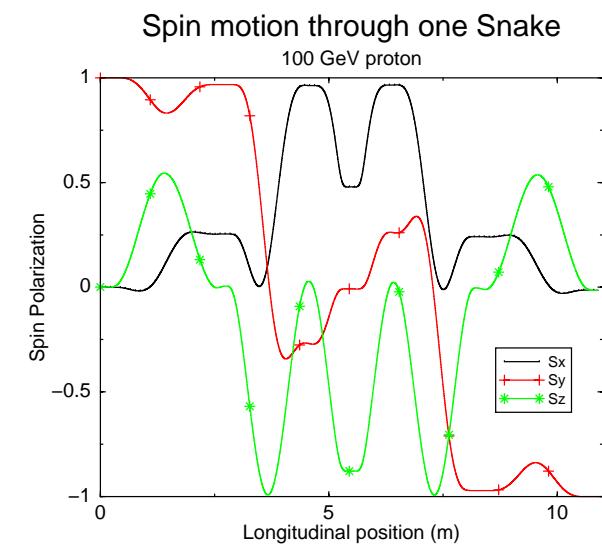
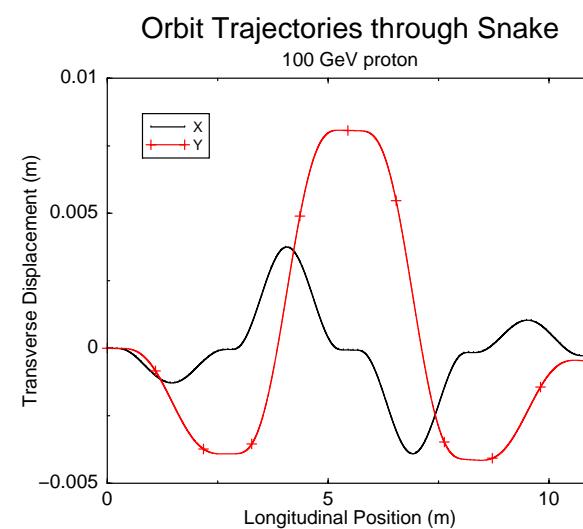
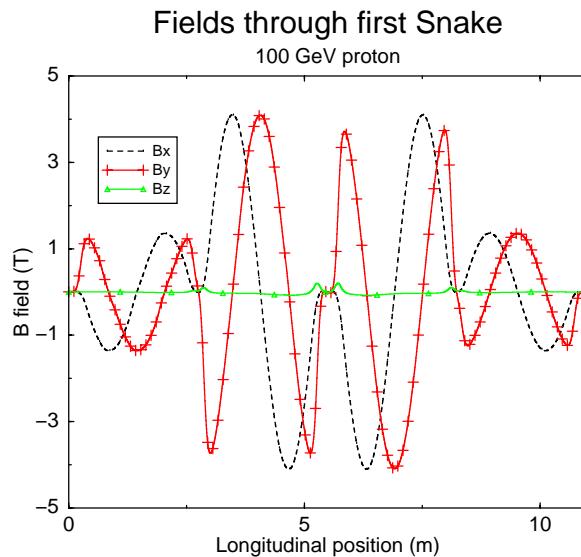
AC dipole used to increase strength  
of  $\nu_s = Q_y$  resonance.

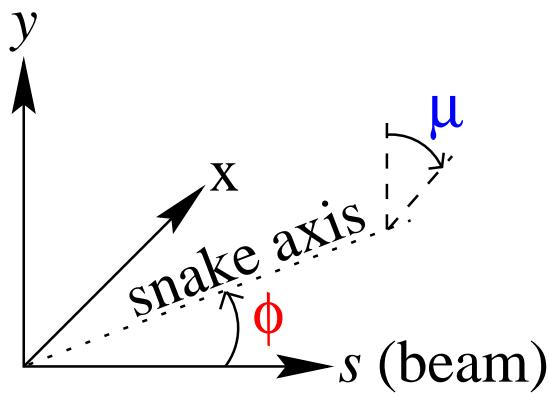
# Comments on Injector Performance

- Source: Worked beautifully.
- Booster: No depolarization.
- AGS: Polarization loss larger in FY02 due to lower ramp rate and higher bunch intensity
  - Failed main magnet power supply.  
(Repair by Fall'02.)
- AGS: New partial superconducting helical snake should give polarization  $\sim 70\%$ .



# \_trajectory and Spin through Snakes

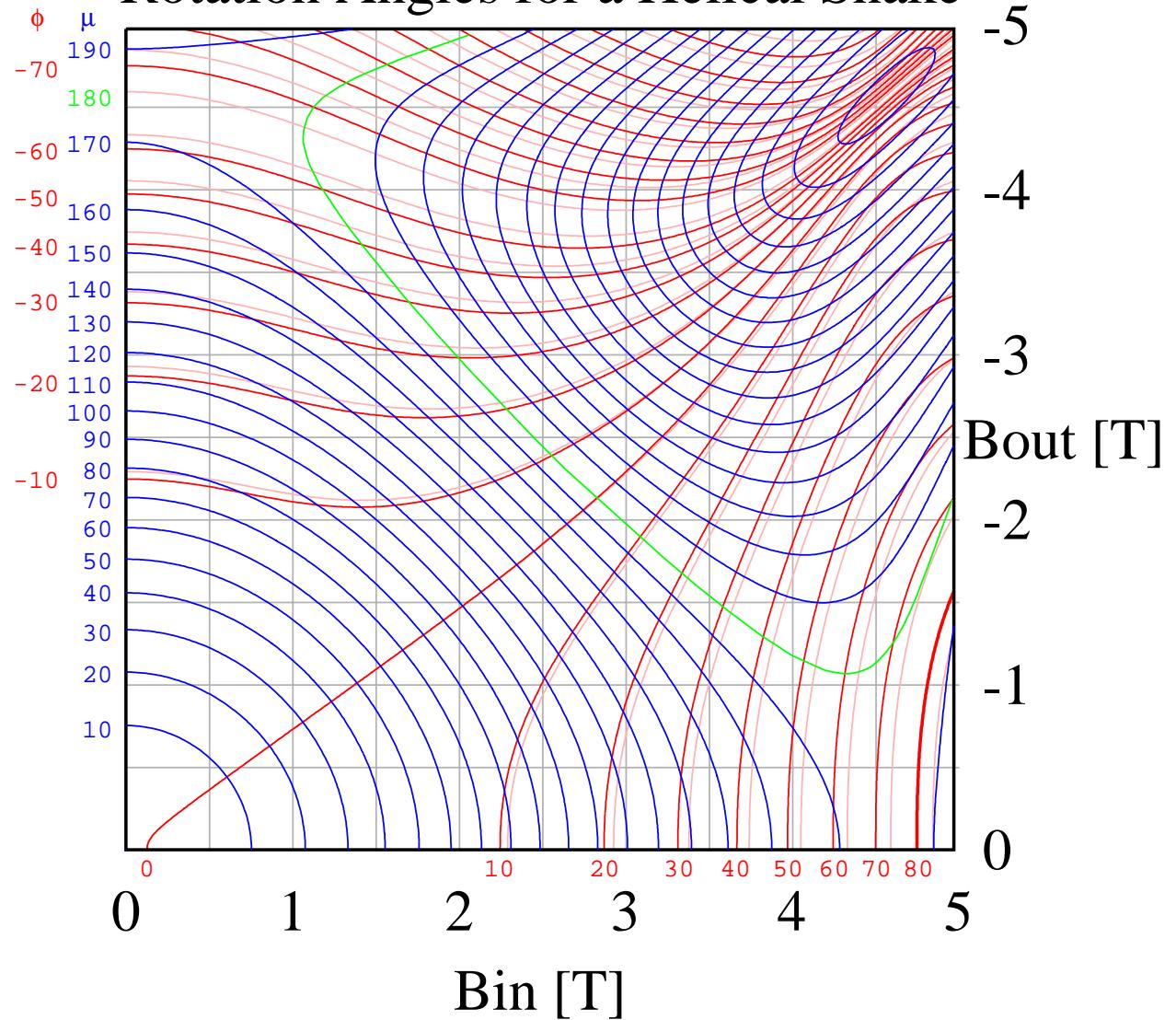




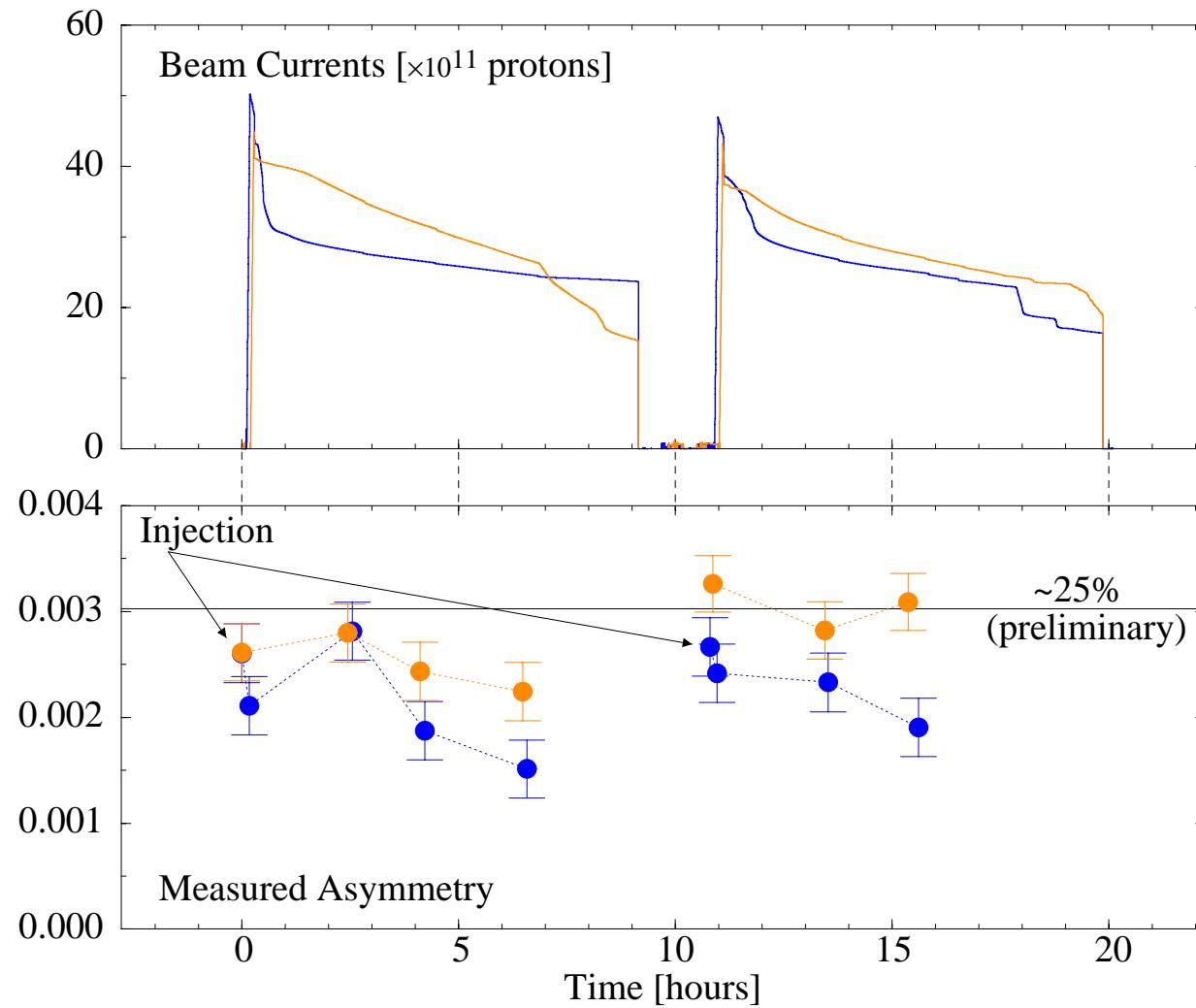
The rotation axis of the snake is  $\phi$ , and  $\mu$  is the rotation angle.

Note that the  $\phi$  contours shift slightly from injection (red) at 25 GeV to storage (pink) at 250 GeV.

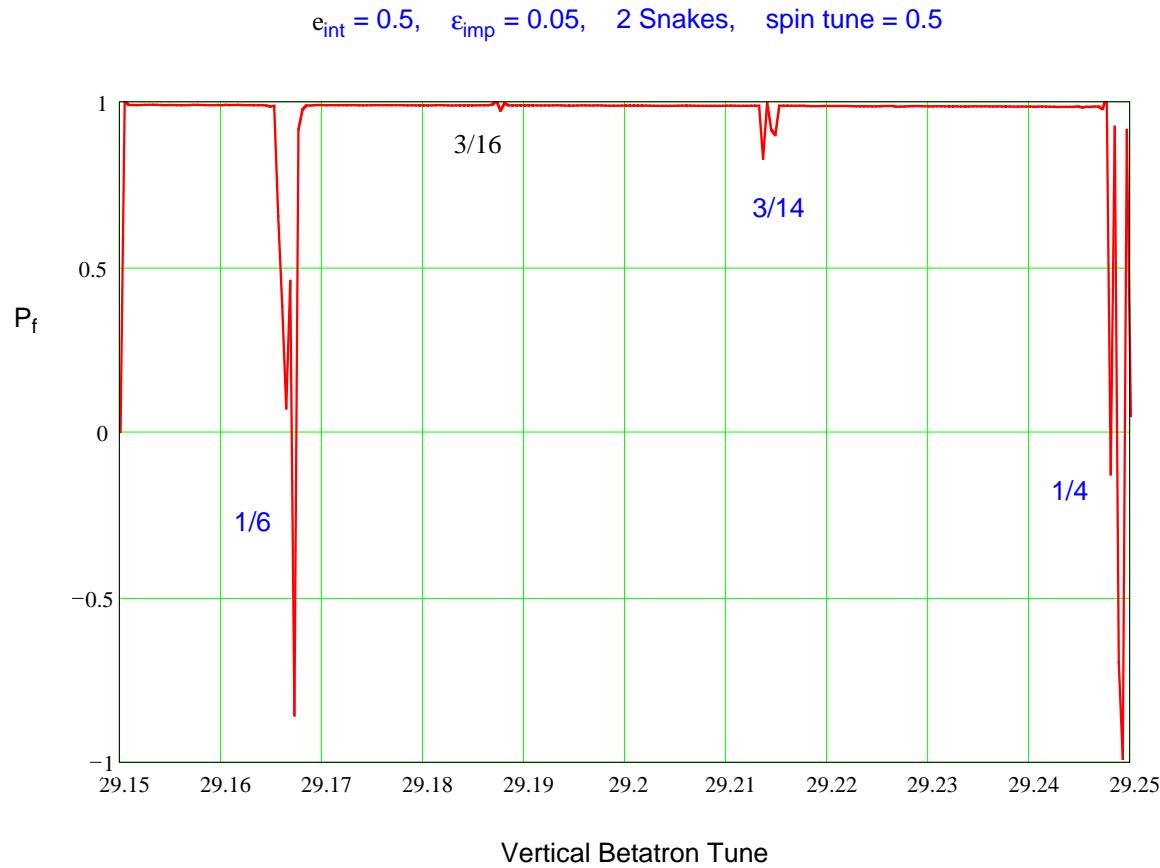
## Rotation Angles for a Helical Snake



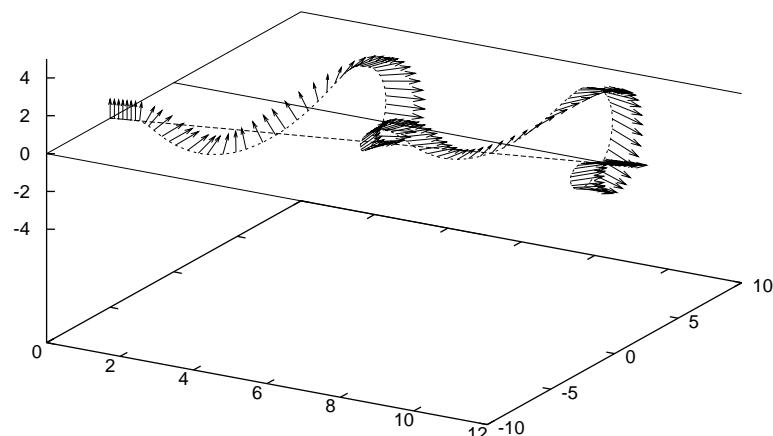
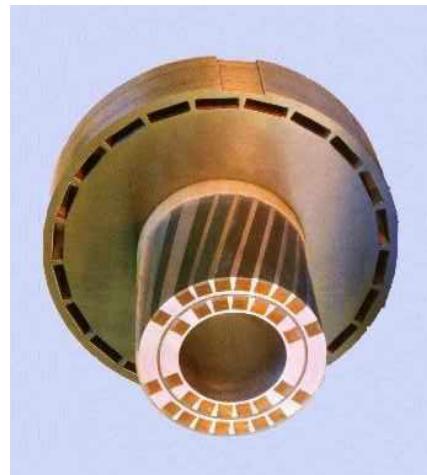
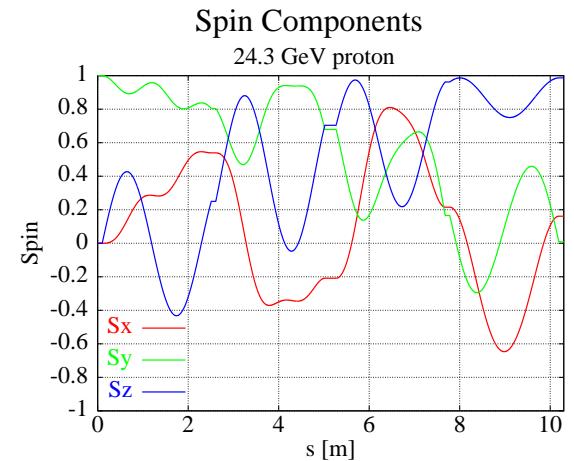
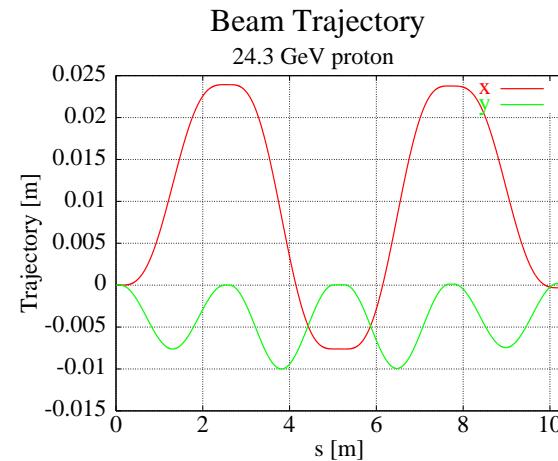
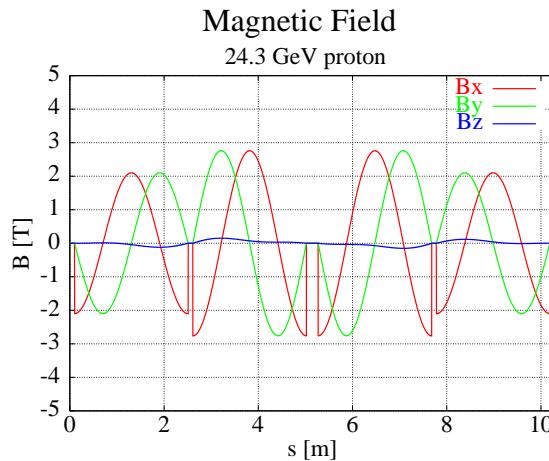
# RHIC Beam Polarization



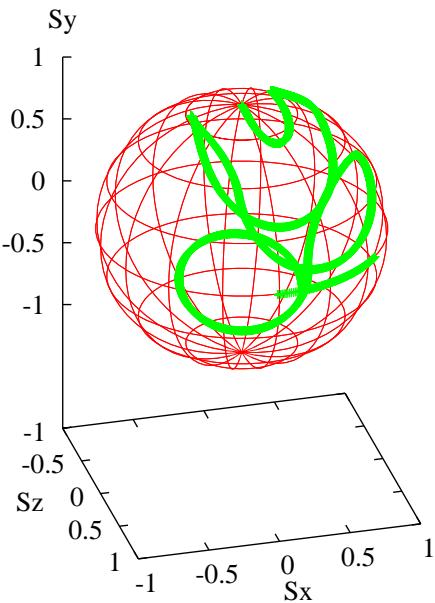
# Snake Resonances



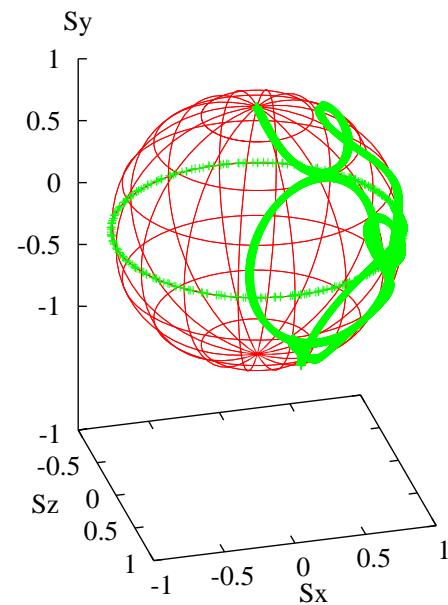
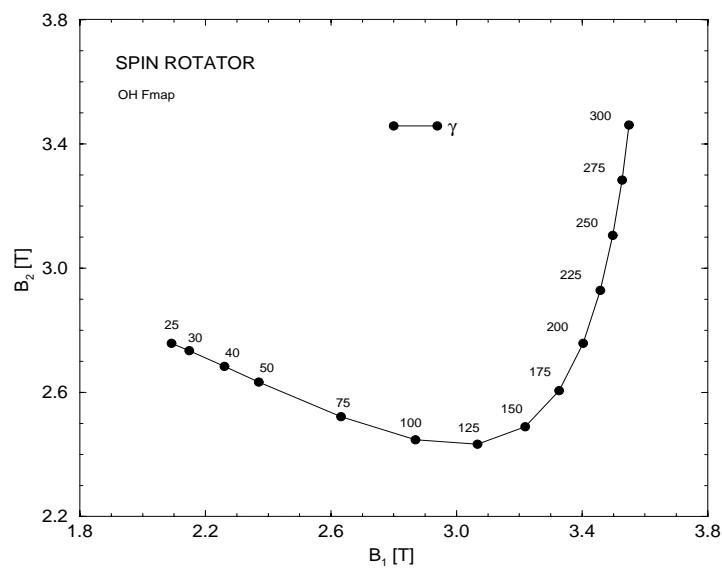
# Helical Spin Rotators



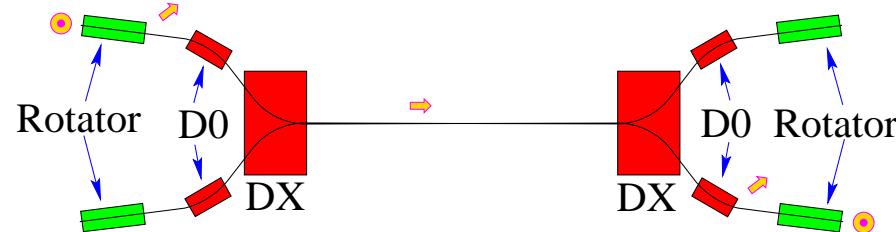
# Compensation for D0-DX Bends

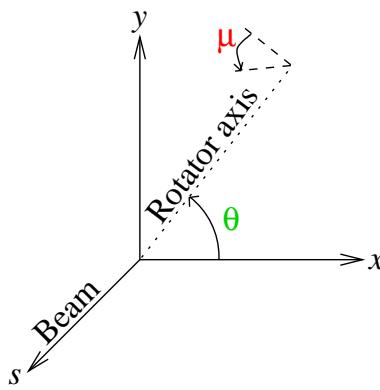


$E = 25 \text{ GeV}$



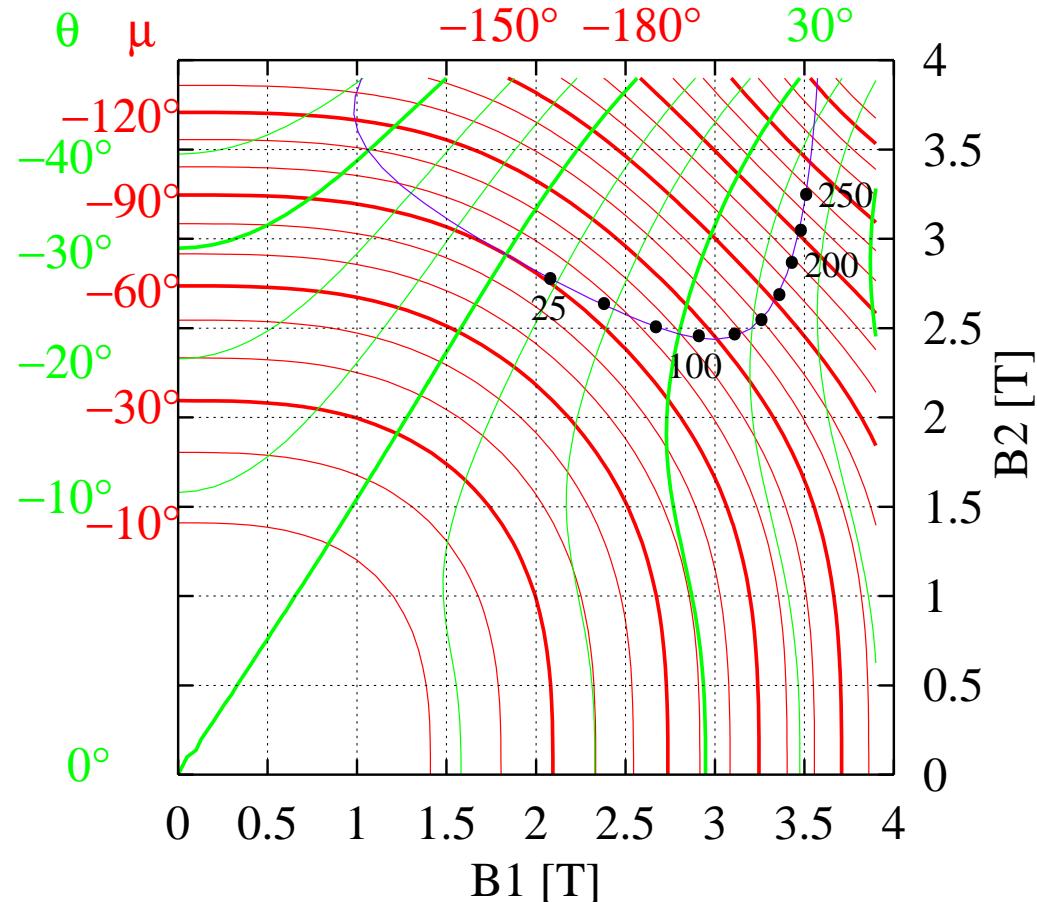
$E = 250 \text{ GeV}$





The rotation axis of the spin rotator is in the  $x$ - $y$  plane at an angle  $\theta$  from the vertical. The spin is rotated by the angle  $\mu$  around the rotation axis.

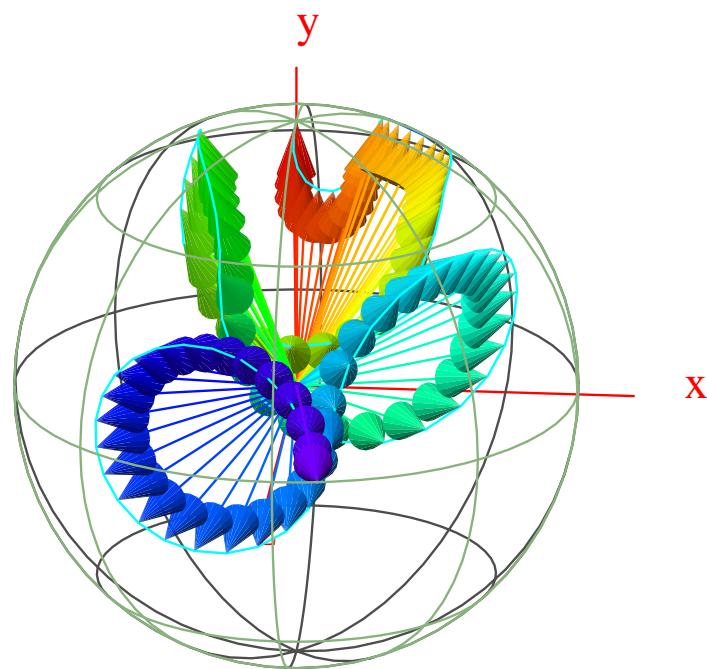
## Rotation Angles for a Helical Spin Rotator



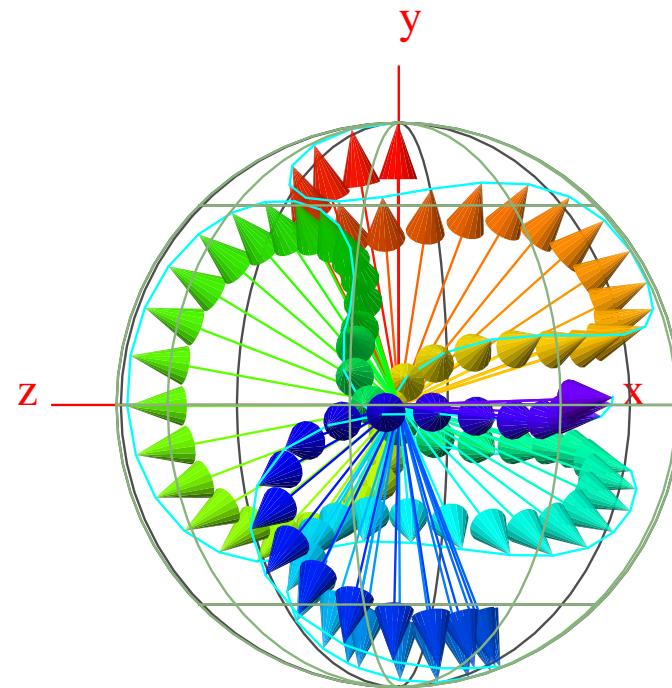
Note: Purple contour for rotation into horizontal plane.  
Black dots show settings for RHIC energies in increments of 25 GeV from 25 to 250 GeV.

# ♪ Rotator Spin Precession ♪

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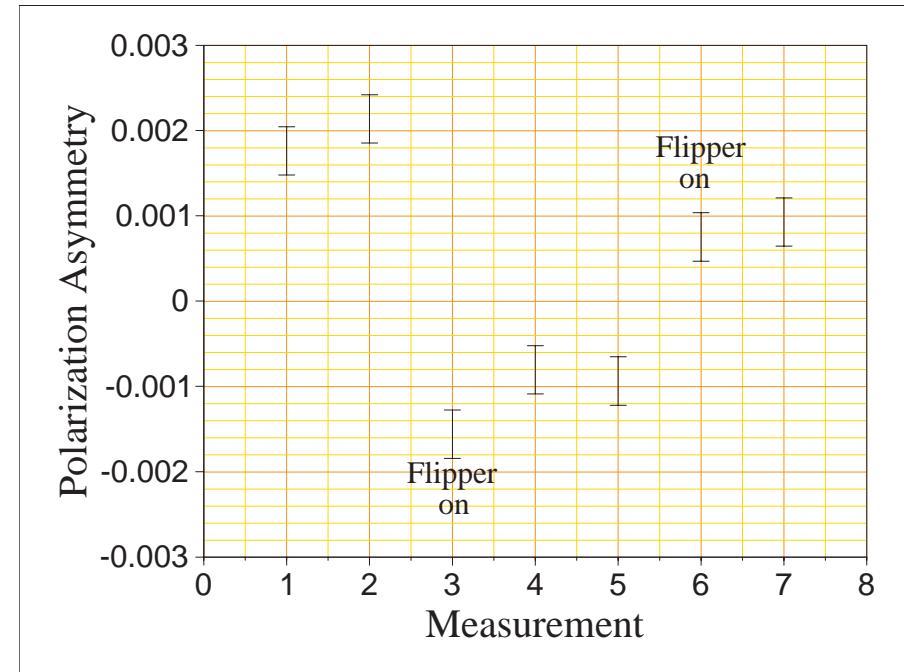


Rotator's spin vector at injection energy



Rotator's spin vector at 250 GeV

# Spin Flipper (AC Dipole)



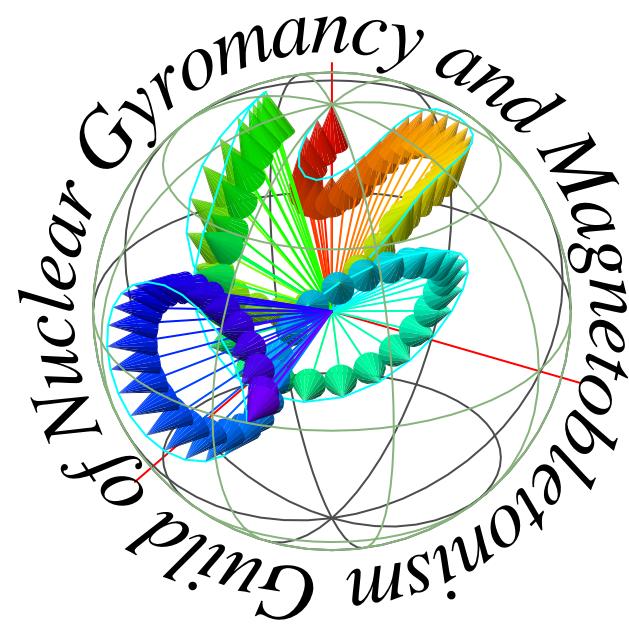
First successful spin flip in Blue ring.

# Some References (by no means all!)

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Requirement for Membership: Can pronounce “nuclear” correctly.